

Contents

Preface	iii
Contents	v
Introduction	
Some questions you may have about this book	1
Some thoughts for teachers	2
	10
Part I Topology, and the Brouwer Fixed-Point Theorem in Two Dimensions	15
1 Getting Started	17
1.1 Sets	19
1.2 Logical statements and notation	31
1.3 Quantification: “for all” and “there exists”	35
1.4 Proving implication and equivalence	40
2 Topological Spaces	49
2.1 Closure operators and topological spaces	55
2.2 Examples of closure operators	58
2.3 Some basic facts, notation, and terminology	63
3 A Somewhat More Formal Introduction	67
3.1 Symbols and mathematical language	69
3.2 Results and their statements	71
3.3 Deduction, proof, and formalization	73
3.4 Foundational and descriptive axioms	77
3.5 Axioms: the good, the bad, and the ugly	80
3.6 Proof by contradiction	83
3.7 The contrapositive	85
4 Some Things You Should Know	89

Contents

4.1	Negation of quantifiers, De Morgan's laws, and truth tables	91
4.2	Complements and truth sets	95
4.3	Safe sets: power sets and cartesian products	98
4.4	Relations	100
4.5	Functions	101
4.6	More on cartesian products	108
4.7	Identification and equivalence relations	110
5	Infinities	117
5.1	Arbitrary unions, intersections, and induction	119
5.2	Small is beautiful	127
5.3	Complements, duality, and open sets	131
5.4	*An alternative characterization of topological spaces, which no one uses	133
5.5	*The customary definition of "topological space"	136
6	Numbers and Their Properties	141
6.1	Properties of the real numbers	150
	Algebraic axioms	151
	Order axioms	153
	The least upper bound property	154
6.2	All of familiar algebra follows from the axioms	156
6.3	Absolute value	161
6.4	The natural numbers, \mathbb{N}	164
6.5	The integers, \mathbb{Z}	171
6.6	The rational numbers, \mathbb{Q}	175
6.7	Definition by induction (recursion)	178
6.8	Intervals	182
6.9	The Archimedean property	186
6.10	Orphan problems	190
7	The Metric Closure Operator	193
7.1	Metric spaces	195
7.2	Distance in \mathbb{R}^n	198
7.3	The triangle inequality for the Euclidean metric	201
7.4	The metric closure operator	207
7.5	Examples and applications	213
7.6	Balls and open sets	217
7.7	Open neighborhoods	220
7.8	*Interior, exterior, and boundary	222
8	More About Functions	227
8.1	Composition of functions	231
8.2	Injections, restrictions, and extensions	232
8.3	Range and surjections	234
8.4	Inverse functions and one-to-one correspondences (bijections)	235
8.5	Functions and order	238

Contents

8.6 Induced set functions	240
9 Continuous Functions	245
9.1 Continuity defined	248
9.2 Another characterization of continuity	252
9.3 Continuity in metric spaces	255
9.4 Continuity and cartesian products	259
9.5 New continuous functions from old	265
10 Completeness of the Real Numbers	275
10.1 The principle of nested closed intervals	281
10.2 The intermediate-value theorem and the bisection method	283
10.3 *The decimal expansion principle	287
10.4 *Equivalent formulations of completeness for \mathbb{R}	293
10.5 *Connectedness	295
10.6 *The Cantor set	302
11 Convergence	309
11.1 Convergence and limits	312
11.2 Subsequences	316
11.3 Sequential compactness	318
11.4 *Infinite series	326
11.5 *Completeness for metric spaces	335
11.6 *The contraction mapping theorem	337
11.7 *Compactness	339
12 Brouwer's Theorem in Two Dimensions	345
12.1 The proof	348
12.2 A combinatorial lemma	350
12.3 Proof of the combinatorial lemma	352
13 Topological Equivalence	359
13.1 Homeomorphisms	362
13.2 Topological invariants	369
13.3 The fixed-point property and conjugacy	370
13.4 $[0, 1]^2 \approx B^2$	374
13.5 *Cantor spaces	377
Part II Foundations	383
14 Propositional Logic	385
14.1 Symbols and logical formulas	386
14.2 Valuations and truth	390
14.3 Proofs	392
14.4 Subproofs	394

Contents

14.5 Indirect proofs and turnstile notation	396
14.6 Soundness of propositional logic	401
14.7 Completeness of propositional logic	403
15 First-Order Logic	407
15.1 Formal theories and axioms	407
15.2 Syntax in first-order logic	414
15.3 Interpretations, Truth Values, and Models	416
15.4 Substitution, and two rules for quantifiers	421
15.5 Loose variables, and the other two rules for quantifiers	423
15.6 Rules for equality	425
15.7 Proofs in first-order logic, and soundness	426
15.8 Completeness of first-order logic	430
16 Axioms for Set Theory	437
16.1 Sets and the axiom of extension	438
16.2 Russell's paradox and the axiom of specification	439
16.3 An interlude on being and nothingness	441
16.4 The pair axiom and ordered pairs	441
16.5 Subsets, set operations, and the union axiom	442
16.6 Power sets, the power set axiom, and cartesian products	445
16.7 Relations and functions	446
16.8 Peano arithmetic and the axiom of infinity	448
16.9 Two "final" axioms	451
16.10 The axiom of choice	454
16.11 Ordinal numbers	455
17 Recursion and Arithmetic	463
17.1 Recursion	463
17.2 The "uniqueness" of \mathbb{N}	468
17.3 Arithmetic in \mathbb{N}	471
17.4 Order in \mathbb{N}	477
18 Finite and Infinite Sets	481
18.1 Finite sets	481
18.2 Counting	484
18.3 Countable sets	489
18.4 Smaller infinity, bigger infinity	492
19 Construction Zone: Numbers	499
19.1 Constructing the non-negative rational numbers	501
19.2 Constructing the non-negative real numbers	506
Dedekind cuts	507
Ordering of Dedekind cuts	509
Addition of Dedekind cuts	510
Multiplication of Dedekind cuts	513

Contents

19.3 Going negative	514
19.4 Other constructions of the real numbers	518
19.5 Uniqueness of the real numbers	519
20 Trouble in Paradise	523
20.1 Crises in the Foundations of Mathematics	523
20.2 Gödel's terrible swift sword	527
20.3 The halting problem and incompleteness	530
20.4 An idealized computer	532
20.5 Gödel-Rosser, and consistency	537
20.6 Other things we can't prove	539
20.7 Valedictory	541
Appendices	543
A Things Mathematicians Do	545
B Letters and Symbols	549
Index	555