Contents

Preface iii

Contents v

Introduction 1

Some questions you may have about this book 2
Some thoughts for teachers 10

Part I Topology, and the Brouwer Fixed-Point Theorem in Two Dimensions 15

1 Getting Started 17

1.1 Sets 19
1.2 Logical statements and notation 32
1.3 Quantification: “for all” and “there exists” 39
1.4 Proving implication and equivalence 44

2 Topological Spaces 53

2.1 Closure operators and topological spaces 59
2.2 Examples of closure operators 62
2.3 Some basic facts, notation, and terminology 67

3 A Somewhat More Formal Introduction 71

3.1 Symbols and mathematical language 73
3.2 Results and their statements 75
3.3 Deduction, proof, and formalization 77
3.4 Foundational and descriptive axioms 81
3.5 Axioms: the good, the bad, and the ugly 85
3.6 Proof by contradiction 87
3.7 The contrapositive 90

4 Some Things You Should Know 93

DRAFT not for circulation, distribution, or quotation
© 2021–2025 Daniel Goroff / Nat Kuhn. Compiled 8/19/2025 at 10:20:00, v0.12.3

4.1 Negation of quantifiers, De Morgan's laws, and truth tables . . . 95
4.2 Complements and truth sets . . . 99
4.3 Safe sets: power sets and cartesian products . . . 102
4.4 Relations . . . 104
4.5 Functions . . . 106
4.6 More on cartesian products . . . 112
4.7 Identification and equivalence relations . . . 114

5 Infinities **121**
5.1 Arbitrary unions, intersections, and induction . . . 123
5.2 Small is beautiful . . . 131
5.3 Complements, duality, and open sets . . . 135
5.4 *An alternative characterization of topological spaces, which no one uses 137
5.5 *The customary definition of "topological space" . . . 140

6 Numbers and Their Properties **145**
6.1 Properties of the real numbers . . . 154
Algebraic axioms . . . 155
Order axioms . . . 157
The least upper bound property . . . 158
6.2 All of familiar algebra follows from the axioms . . . 160
6.3 Absolute value . . . 165
6.4 The natural numbers, $\mathbb{N}$ . . . 168
6.5 The integers, $\mathbb{Z}$ . . . 175
6.6 The rational numbers, $\mathbb{Q}$ . . . 179
6.7 Definition by induction (recursion) . . . 182
6.8 Intervals . . . 186
6.9 The Archimedean property . . . 190
6.10 Orphan problems . . . 194

7 The Metric Closure Operator **197**
7.1 Metric spaces . . . 199
7.2 Distance in $\mathbb{R}^n$ . . . 202
7.3 The triangle inequality for the Euclidean metric . . . 205
7.4 The metric closure operator . . . 211
7.5 Examples and applications . . . 217
7.6 Balls and open sets . . . 221
7.7 Open neighborhoods . . . 224
7.8 *Interior, exterior, and boundary . . . 226

8 More About Functions **231**
8.1 Composition of functions . . . 235
8.2 Injections, restrictions, and extensions . . . 236
8.3 Range and surjections . . . 238
8.4 Inverse functions and one-to-one correspondences (bijections) . . . 239
8.5 Functions and order . . . 243

DRAFT not for circulation, distribution, or quotation
© 2021–2025 Daniel Goroff / Nat Kuhn. Compiled 8/19/2025 at 10:20:00, v0.12.3

8.6 Induced set functions . . . 244

9 Continuous Functions 249

9.1 Continuity defined . . . 252
9.2 Another characterization of continuity . . . 256
9.3 Continuity in metric spaces . . . 259
9.4 Continuity and cartesian products . . . 263
9.5 New continuous functions from old . . . 269

10 Completeness of the Real Numbers 279

10.1 The principle of nested closed intervals . . . 285
10.2 The intermediate-value theorem and the bisection method . . . 287
10.3 *The decimal expansion principle . . . 291
10.4 *Equivalent formulations of completeness for $\mathbb{R}$ . . . 297
10.5 *Connectedness . . . 299
10.6 *The Cantor set . . . 306

11 Convergence 313

11.1 Convergence and limits . . . 316
11.2 Subsequences . . . 320
11.3 Sequential compactness . . . 322
11.4 *Infinite series . . . 330
11.5 *Completeness for metric spaces . . . 339
11.6 *The contraction mapping theorem . . . 341
11.7 *Compactness . . . 343

12 Brouwer's Theorem in Two Dimensions 349

12.1 The proof . . . 352
12.2 A combinatorial lemma . . . 354
12.3 Proof of the combinatorial lemma . . . 356

13 Topological Equivalence 363

13.1 Homeomorphisms . . . 366
13.2 Topological invariants . . . 373
13.3 The fixed-point property and conjugacy . . . 374
13.4 $[0,1]^2 \approx B^2$ . . . 378
13.5 *Cantor spaces . . . 381

Part II Foundations 387

14 Propositional Logic 389

14.1 Symbols and logical formulas . . . 390
14.2 Valuations and truth . . . 394
14.3 Proofs . . . 396
14.4 Subproofs . . . 398

DRAFT not for circulation, distribution, or quotation
© 2021–2025 Daniel Goroff / Nat Kuhn. Compiled 8/19/2025 at 10:20:00, v0.12.3

14.5 Indirect proofs and turnstile notation . . . 400
14.6 Soundness of propositional logic . . . 405
14.7 Completeness of propositional logic . . . 407

15 First-Order Logic **411**

15.1 Formal theories and axioms . . . 411
15.2 Syntax in first-order logic . . . 418
15.3 Interpretations, Truth Values, and Models . . . 420
15.4 Substitution, and two rules for quantifiers . . . 425
15.5 Loose variables, and the other two rules for quantifiers . . . 427
15.6 Rules for equality . . . 429
15.7 Proofs in first-order logic, and soundness . . . 430
15.8 Completeness of first-order logic . . . 434

16 Axioms for Set Theory **441**

16.1 Sets and the axiom of extension . . . 442
16.2 Russell's paradox and the axiom of specification . . . 443
16.3 An interlude on being and nothingness . . . 445
16.4 The pair axiom and ordered pairs . . . 445
16.5 Subsets, set operations, and the union axiom . . . 446
16.6 Power sets, the power set axiom, and cartesian products . . . 449
16.7 Relations and functions . . . 450
16.8 Peano arithmetic and the axiom of infinity . . . 452
16.9 Two "final" axioms . . . 455
16.10 The axiom of choice . . . 458
16.11 Ordinal numbers . . . 460

17 Recursion and Arithmetic **467**

17.1 Recursion . . . 467
17.2 The "uniqueness" of $\mathbb{N}$ . . . 472
17.3 Arithmetic in $\mathbb{N}$ . . . 475
17.4 Order in $\mathbb{N}$ . . . 481

18 Finite and Infinite Sets **485**

18.1 Finite sets . . . 485
18.2 Counting . . . 488
18.3 Countable sets . . . 493
18.4 Smaller infinity, bigger infinity . . . 496

19 Construction Zone: Numbers **503**

19.1 Constructing the non-negative rational numbers . . . 505
19.2 Constructing the non-negative real numbers . . . 510
Dedekind cuts . . . 511
Ordering of Dedekind cuts . . . 513
Addition of Dedekind cuts . . . 514
Multiplication of Dedekind cuts . . . 517

DRAFT not for circulation, distribution, or quotation
© 2021–2025 Daniel Goroff / Nat Kuhn. Compiled 8/19/2025 at 10:20:00, v0.12.3

19.3 Going negative . . . 518
19.4 Other constructions of the real numbers . . . 522
19.5 Uniqueness of the real numbers . . . 523

20 Trouble in Paradise **527**

20.1 Crises in the Foundations of Mathematics . . . 527
20.2 Gödel's terrible swift sword . . . 531
20.3 The halting problem and incompleteness . . . 534
20.4 An idealized computer . . . 536
20.5 Gödel-Rosser, and consistency . . . 541
20.6 Other things we can't prove . . . 543
20.7 Valedictory . . . 545

Appendices **547**

A Things Mathematicians Do **549**

B Letters and Symbols **553**

Bibliography **559**

Index **561**

DRAFT not for circulation, distribution, or quotation
© 2021–2025 Daniel Goroff / Nat Kuhn. Compiled 8/19/2025 at 10:20:00, v0.12.3